Unitarity Quadrangles of Four Neutrino Mixing

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Abstract

We present a classification of the unitarity quadrangles in the four-neutrino mixing scheme. We find that there are totally thirty-six distinct topologies among twelve different unitarity quadrangles. Concise relations are established between the areas of those unitarity quadrangles and the rephasing invariants of CP and T violation.

PACS number(s): 14.60.Pq, 13.10.+q, 25.30.Pt

The robust Super-Kamiokande [1] and SNO [2] data have provided convincing evidence that the atmospheric ν_μ neutrinos convert primarily into ν_τ neutrinos, while the solar ν_e neutrinos convert essentially into ν_μ or ν_τ neutrinos. It turns out that the existence of a light sterile neutrino ν_s, which has been assumed to reconcile the LSND [3] evidence for $\overline{\nu}_\mu \rightarrow \overline{\nu}_e$ (and $\nu_\mu \rightarrow \nu_e$) oscillations with solar and atmospheric neutrino data in the four-neutrino mixing scenarios [4] [1], becomes questionable. Indeed, a recent globable analysis of current neutrino oscillation data has shown that the well-known (2+2) and (3+1) schemes of four neutrino mixing are both disfavored [7]. The upcoming MiniBooNE experiment [8] is therefore crucial, in order to confirm or disprove the LSND measurement. Before a definitely negative conclusion can be drawn from MiniBooNE, however, the LSND data should be taken seriously. In particular, it is worthwhile to investigate the four-neutrino mixing scenarios in a way without special theoretical biases and (or) empirical assumptions.

In a recent paper [9], we have calculated the rephasing invariants of CP and T violation by use of a favorable or "standard" parametrization of the generic 4×4 neutrino mixing matrix. Our results are expected to be quite useful for a systematical analysis of CP- and T-violating effects in various long-baseline neutrino oscillation experiments. In the present work, which may serve as an important addendum to Ref. [9], we aim to present a complete geometrical description of CP and T violation in the four-neutrino mixing scheme.

It is well known that the language of unitarity triangles is very helpful for the description of CP violation in the quark sector [10]. The same language has been introduced into the lepton sector to describe CP violation in the framework of three-family lepton flavor mixing [11–14]. In Ref. [12], an example concerning the unitarity quadrangles is given to illustrate the necessary condition of CP violation in the four-neutrino mixing scenario. In this Brief Report, we shall first make a classification of all possible unitarity quadrangles and then calculate their areas in terms of the rephasing invariants of CP violation based on the standard parametrization of four neutrino mixing.

Let us consider the admixture of three active $(\nu_e, \nu_\mu, \nu_\tau)$ neutrinos and one sterile (ν_s) neutrino. Although ν_s does not participate in any normal weak interactions, it may oscillate with ν_e, ν_μ and ν_τ [15]. Hence a 4×4 unitary matrix V is required to fully describe four neutrino mixing in neutrino oscillations. In the basis where the flavor and mass eigenstates of charged leptons are identical, V is defined to link the neutrino mass eigenstates $(\nu_0, \nu_1, \nu_2, \nu_3)$ to the neutrino flavor eigenstates $(\nu_s, \nu_e, \nu_\mu, \nu_\tau)$:

$$\begin{pmatrix} \nu_s \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{s0} & V_{s1} & V_{s2} & V_{s3} \\ V_{e0} & V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 0} & V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 0} & V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} . \tag{1}$$

As neutrinos are expected to be Majorana particles, a full parametrization of V needs six mixing angles and six CP-violating phases. Here we make use of the standard parametrization advocated in Ref. [16]; i.e.,

[1] Instead of introducing a light sterile neutrino, a few more far-fetched ideas (such as the violation of CPT symmetry in the neutrino sector [5] and the lepton-number-violating muon decay [6]) have been proposed in the literature.

$$
V = \begin{pmatrix}
c_{01}c_{02}c_{03} & c_{02}c_{03}\hat{s}^*_{01} & c_{03}\hat{s}^*_{02} & \hat{s}^*_{03} \\
\begin{matrix} -c_{01}c_{02}\hat{s}_{03}\hat{s}^*_{13} \\ -c_{01}c_{13}\hat{s}_{02}\hat{s}^*_{12} \\ -c_{12}c_{13}\hat{s}_{01} \end{matrix} &
\begin{matrix} -c_{02}\hat{s}^*_{01}\hat{s}_{03}\hat{s}^*_{13} \\ -c_{13}\hat{s}^*_{01}\hat{s}_{02}\hat{s}^*_{12} \\ +c_{01}c_{12}c_{13} \end{matrix} &
\begin{matrix} -\hat{s}^*_{02}\hat{s}_{03}\hat{s}^*_{13} \\ +c_{02}c_{13}\hat{s}^*_{12} \end{matrix} &
c_{03}\hat{s}^*_{13} \\
\begin{matrix} -c_{01}c_{02}c_{13}\hat{s}_{03}\hat{s}^*_{23} \\ +c_{01}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} \\ -c_{01}c_{12}c_{23}\hat{s}_{02} \\ +c_{12}\hat{s}_{01}\hat{s}_{13}\hat{s}^*_{23} \\ +c_{23}\hat{s}_{01}\hat{s}_{12} \end{matrix} &
\begin{matrix} -c_{02}c_{13}\hat{s}^*_{01}\hat{s}_{03}\hat{s}^*_{23} \\ +\hat{s}^*_{01}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} \\ -c_{12}c_{23}\hat{s}^*_{01}\hat{s}_{02} \\ -c_{01}c_{12}\hat{s}_{13}\hat{s}^*_{23} \\ -c_{01}c_{23}\hat{s}_{12} \end{matrix} &
\begin{matrix} -c_{13}\hat{s}^*_{02}\hat{s}_{03}\hat{s}^*_{23} \\ -c_{02}\hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} \\ +c_{02}c_{12}c_{23} \end{matrix} &
c_{03}c_{13}\hat{s}^*_{23} \\
\begin{matrix} -c_{01}c_{02}c_{13}c_{23}\hat{s}_{03} \\ +c_{01}c_{23}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13} \\ +c_{01}c_{12}\hat{s}_{02}\hat{s}_{23} \\ +c_{12}c_{23}\hat{s}_{01}\hat{s}_{13} \\ -\hat{s}_{01}\hat{s}_{12}\hat{s}_{23} \end{matrix} &
\begin{matrix} -c_{02}c_{13}c_{23}\hat{s}^*_{01}\hat{s}_{03} \\ +c_{23}\hat{s}^*_{01}\hat{s}_{02}\hat{s}^*_{12}\hat{s}_{13} \\ +c_{12}\hat{s}^*_{01}\hat{s}_{02}\hat{s}_{23} \\ -c_{01}c_{12}c_{23}\hat{s}_{13} \\ +c_{01}\hat{s}_{12}\hat{s}_{23} \end{matrix} &
\begin{matrix} -c_{13}c_{23}\hat{s}^*_{02}\hat{s}_{03} \\ -c_{02}c_{23}\hat{s}^*_{12}\hat{s}_{13} \\ -c_{02}c_{12}\hat{s}_{23} \end{matrix} &
c_{03}c_{13}c_{23}
\end{pmatrix} , \qquad (2)
$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $\hat{s}_{ij} \equiv s_{ij} e^{{\rm i}\delta_{ij}}$ with $s_{ij} \equiv \sin\theta_{ij}$. Note that only three independent combinations of the six CP-violating phases in V are relevant to CP or T violation in neutrino oscillations [9], whose magnitude is governed by the rephasing invariants $J^{ij}_{\alpha\beta} \equiv {\rm Im}(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i})$ with the Greek subscripts running over (s, e, μ, τ) and the Latin superscripts running over $(0, 1, 2, 3)$.

The unitarity of V implies that there are twelve orthogonality relations and eight normalization conditions among its sixteen matrix elements. The former corresponds to twelve quadrangles in the complex plane, the so-called unitarity quadrangles. To be explicit, let us write out the twelve orthogonality relations and name their corresponding quadrangles:

$$
\begin{aligned}
{\rm Q}_{se}: &\;\; V_{s0}V^*_{e0} + V_{s1}V^*_{e1} + V_{s2}V^*_{e2} + V_{s3}V^*_{e3} = 0 \; , \\
{\rm Q}_{s\mu}: &\;\; V_{s0}V^*_{\mu 0} + V_{s1}V^*_{\mu 1} + V_{s2}V^*_{\mu 2} + V_{s3}V^*_{\mu 3} = 0 \; , \\
{\rm Q}_{s\tau}: &\;\; V_{s0}V^*_{\tau 0} + V_{s1}V^*_{\tau 1} + V_{s2}V^*_{\tau 2} + V_{s3}V^*_{\tau 3} = 0 \; , \\
{\rm Q}_{e\mu}: &\;\; V_{e0}V^*_{\mu 0} + V_{e1}V^*_{\mu 1} + V_{e2}V^*_{\mu 2} + V_{e3}V^*_{\mu 3} = 0 \; , \\
{\rm Q}_{e\tau}: &\;\; V_{e0}V^*_{\tau 0} + V_{e1}V^*_{\tau 1} + V_{e2}V^*_{\tau 2} + V_{e3}V^*_{\tau 3} = 0 \; , \\
{\rm Q}_{\mu\tau}: &\;\; V_{\mu 0}V^*_{\tau 0} + V_{\mu 1}V^*_{\tau 1} + V_{\mu 2}V^*_{\tau 2} + V_{\mu 3}V^*_{\tau 3} = 0 \; ;
\end{aligned} \qquad (3)
$$

and

$$
\begin{aligned}
{\rm Q}_{01}: &\;\; V_{s0}V^*_{s1} + V_{e0}V^*_{e1} + V_{\mu 0}V^*_{\mu 1} + V_{\tau 0}V^*_{\tau 1} = 0 \; , \\
{\rm Q}_{02}: &\;\; V_{s0}V^*_{s2} + V_{e0}V^*_{e2} + V_{\mu 0}V^*_{\mu 2} + V_{\tau 0}V^*_{\tau 2} = 0 \; , \\
{\rm Q}_{03}: &\;\; V_{s0}V^*_{s3} + V_{e0}V^*_{e3} + V_{\mu 0}V^*_{\mu 3} + V_{\tau 0}V^*_{\tau 3} = 0 \; , \\
{\rm Q}_{12}: &\;\; V_{s1}V^*_{s2} + V_{e1}V^*_{e2} + V_{\mu 1}V^*_{\mu 2} + V_{\tau 1}V^*_{\tau 2} = 0 \; , \\
{\rm Q}_{13}: &\;\; V_{s1}V^*_{s3} + V_{e1}V^*_{e3} + V_{\mu 1}V^*_{\mu 3} + V_{\tau 1}V^*_{\tau 3} = 0 \; , \\
{\rm Q}_{23}: &\;\; V_{s2}V^*_{s3} + V_{e2}V^*_{e3} + V_{\mu 2}V^*_{\mu 3} + V_{\tau 2}V^*_{\tau 3} = 0 \; .
\end{aligned} \qquad (4)
$$

If six mixing angles and six CP-violating phases of V are all known, one can plot twelve unitarity quadrangles without ambiguities. Note, however, that each quadrangle has three distinct topologies in the complex plane. For illustration, we take quadrangle Q_{se} for example and show its three topologies in FIG. 1, where the sizes and phases of $V_{si}V_{ei}^*$ (for $i = 0, 1, 2, 3$) have been fixed. One can see that different topologies of quadrangle Q_{se} arise from different orderings of its four sides, and their areas are apparently different from one another. As a whole, there are totally thirty-six different topologies among twelve unitarity quadrangles.

Now we calculate the areas of all unitarity triangles and relate them to the rephasing invariants of CP violation $J^{ij}_{\alpha\beta}$. Taking quadrangle Q_{se} as an example again, we find that the areas of its three distinct topologies can be given respectively by [2]

$$\begin{aligned}
S^a_{se} &= \frac{1}{4}\left[\mathrm{Im}\left(V_{s1}V_{e0}V^*_{s0}V^*_{e1}\right) + \mathrm{Im}\left(V_{s2}V_{e1}V^*_{s1}V^*_{e2}\right) + \mathrm{Im}\left(V_{s3}V_{e2}V^*_{s2}V^*_{e3}\right) + \mathrm{Im}\left(V_{s0}V_{e3}V^*_{s3}V^*_{e0}\right)\right] \\
&= \frac{1}{4}\left(J^{10}_{se} + J^{21}_{se} + J^{32}_{se} + J^{03}_{se}\right) , \\
S^b_{se} &= \frac{1}{4}\left[\mathrm{Im}\left(V_{s2}V_{e0}V^*_{s0}V^*_{e2}\right) + \mathrm{Im}\left(V_{s1}V_{e2}V^*_{s2}V^*_{e1}\right) + \mathrm{Im}\left(V_{s3}V_{e1}V^*_{s1}V^*_{e3}\right) + \mathrm{Im}\left(V_{s0}V_{e3}V^*_{s3}V^*_{e0}\right)\right] \\
&= \frac{1}{4}\left(J^{20}_{se} + J^{12}_{se} + J^{31}_{se} + J^{03}_{se}\right) , \\
S^c_{se} &= \frac{1}{4}\left[\mathrm{Im}\left(V_{s1}V_{e0}V^*_{s0}V^*_{e1}\right) + \mathrm{Im}\left(V_{s3}V_{e1}V^*_{s1}V^*_{e3}\right) + \mathrm{Im}\left(V_{s2}V_{e3}V^*_{s3}V^*_{e2}\right) + \mathrm{Im}\left(V_{s0}V_{e2}V^*_{s2}V^*_{e0}\right)\right] \\
&= \frac{1}{4}\left(J^{10}_{se} + J^{31}_{se} + J^{23}_{se} + J^{02}_{se}\right) , \qquad (5)
\end{aligned}$$

where J^{ij}_{se} have been defined below Eq. (2). In a similar way, one may calculate the areas of the other eleven unitarity quadrangles. The results for thirty-six different topologies of twelve unitarity quadrangles are summarized as follows:

$$\begin{aligned}
S^a_{\alpha\beta} &= \frac{1}{4}\left(J^{10}_{\alpha\beta} + J^{21}_{\alpha\beta} + J^{32}_{\alpha\beta} + J^{03}_{\alpha\beta}\right) , \\
S^a_{ij} &= \frac{1}{4}\left(J^{ij}_{es} + J^{ij}_{\mu e} + J^{ij}_{\tau\mu} + J^{ij}_{s\tau}\right) ; \qquad (6)
\end{aligned}$$

$$\begin{aligned}
S^b_{\alpha\beta} &= \frac{1}{4}\left(J^{20}_{\alpha\beta} + J^{12}_{\alpha\beta} + J^{31}_{\alpha\beta} + J^{03}_{\alpha\beta}\right) , \\
S^b_{ij} &= \frac{1}{4}\left(J^{ij}_{\mu s} + J^{ij}_{e\mu} + J^{ij}_{\tau e} + J^{ij}_{s\tau}\right) ; \qquad (7)
\end{aligned}$$

and

[2] It should be noted that the areas of unitarity quadrangles under discussion are "algebraic areas", namely, they can be either positive or negative. Of course, it is always possible to take $S^a_{se} = (|J^{10}_{se}| + |J^{21}_{se}| + |J^{32}_{se}| + |J^{03}_{se}|)/4$ or $S^a_{se} = |J^{10}_{se} + J^{21}_{se} + J^{32}_{se} + J^{03}_{se}|/4$, such that S^a_{se} is definitely positive. We find, however, that the language of "algebraic areas" is simpler and more convenient in the description of unitarty quadrangles, which essentially reflect the existence of CP violation the leptonic sector.

$$S^c_{\alpha\beta} = \frac{1}{4}\left(J^{10}_{\alpha\beta} + J^{31}_{\alpha\beta} + J^{23}_{\alpha\beta} + J^{02}_{\alpha\beta}\right) \; ,$$
$$S^c_{ij} = \frac{1}{4}\left(J^{ij}_{es} + J^{ij}_{\tau e} + J^{ij}_{\mu\tau} + J^{ij}_{s\mu}\right) \; , \tag{8}$$

where the subscripts $\alpha\beta = se$, $s\mu$, $s\tau$, $e\mu$, $e\tau$ or $\mu\tau$, and $ij = 01$, 02, 03, 12, 13 or 23. Inversely, one may express $J^{ij}_{\alpha\beta}$ in terms of $S^q_{\alpha\beta}$ or S^q_{ij}. The explicit formulas are

$$\begin{pmatrix} J^{01}_{\alpha\beta} \\ J^{02}_{\alpha\beta} \\ J^{03}_{\alpha\beta} \\ J^{12}_{\alpha\beta} \\ J^{13}_{\alpha\beta} \\ J^{23}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} S^a_{\alpha\beta} \\ S^b_{\alpha\beta} \\ S^c_{\alpha\beta} \end{pmatrix} \; , \tag{9}$$

which corresponds to the unitarity quadrangles Q_{se}, $\mathrm{Q}_{s\mu}$, $\mathrm{Q}_{s\tau}$, $\mathrm{Q}_{e\mu}$, $\mathrm{Q}_{e\tau}$ and $\mathrm{Q}_{\mu\tau}$ defined in Eq. (3); and

$$\begin{pmatrix} J^{ij}_{se} \\ J^{ij}_{s\mu} \\ J^{ij}_{s\tau} \\ J^{ij}_{e\mu} \\ J^{ij}_{e\tau} \\ J^{ij}_{\mu\tau} \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} S^a_{ij} \\ S^b_{ij} \\ S^c_{ij} \end{pmatrix} \; , \tag{10}$$

which corresponds to the unitarity quadrangles Q_{01}, Q_{02}, Q_{03}, Q_{12}, Q_{13} and Q_{23} defined in Eq. (4).

As $J^{ij}_{\alpha\beta} = -J^{ji}_{\alpha\beta} = -J^{ij}_{\beta\alpha} = J^{ji}_{\beta\alpha}$ holds by definition, one may easily obtain $S^q_{\alpha\beta} = -S^q_{\beta\alpha}$ and $S^q_{ij} = -S^q_{ji}$ (for $q = a$, b, c). From the sum rule [9,16]

$$\sum_i J^{ij}_{\alpha\beta} \; = \; \sum_j J^{ij}_{\alpha\beta} \; = \; \sum_\alpha J^{ij}_{\alpha\beta} \; = \; \sum_\beta J^{ij}_{\alpha\beta} \; = \; 0 \; , \tag{11}$$

one can also find

$$\sum_\alpha S^q_{\alpha\beta} \; = \; \sum_\beta S^q_{\alpha\beta} \; = \; \sum_i S^q_{ij} \; = \; \sum_j S^q_{ij} \; = \; 0 \; , \tag{12}$$

where α or β runs over (s, e, μ, τ), and i or j runs over $(0, 1, 2, 3)$. In addition to Eq. (12), the following relations can be derived from Eqs. (6), (7) and (8):

$$
\begin{aligned}
-S^a_{se} - S^a_{e\mu} - S^a_{\mu\tau} + S^a_{s\tau} &= -S^a_{01} - S^a_{12} - S^a_{23} + S^a_{03} \; , \\
-S^a_{s\mu} + S^a_{e\mu} - S^a_{e\tau} + S^a_{s\tau} &= -S^b_{01} - S^b_{12} - S^b_{23} + S^b_{03} \; , \\
-S^a_{se} - S^a_{e\tau} + S^a_{\mu\tau} + S^a_{s\mu} &= -S^c_{01} - S^c_{12} - S^c_{23} + S^c_{03} \; ;
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
-S^b_{se} - S^b_{e\mu} - S^b_{\mu\tau} + S^b_{s\tau} &= -S^a_{02} + S^a_{12} - S^a_{13} + S^a_{03} \; , \\
-S^b_{s\mu} + S^b_{e\mu} - S^b_{e\tau} + S^b_{s\tau} &= -S^b_{02} + S^b_{12} - S^b_{13} + S^b_{03} \; , \\
-S^b_{se} - S^b_{e\tau} + S^b_{\mu\tau} + S^b_{s\mu} &= -S^c_{02} + S^c_{12} - S^c_{13} + S^c_{03} \; ;
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
-S^c_{se} - S^c_{e\mu} - S^c_{\mu\tau} + S^c_{s\tau} &= -S^a_{01} - S^a_{13} + S^a_{23} + S^a_{02} \; , \\
-S^c_{s\mu} + S^c_{e\mu} - S^c_{e\tau} + S^c_{s\tau} &= -S^b_{01} - S^b_{13} + S^b_{23} + S^b_{02} \; , \\
-S^c_{se} - S^c_{e\tau} + S^c_{\mu\tau} + S^c_{s\mu} &= -S^c_{01} - S^c_{13} + S^c_{23} + S^c_{02} \; .
\end{aligned}
\tag{15}
$$

The correlation equations (12) – (15) indicate that there are only nine independent $S^q_{\alpha\beta}$ and (or) S^q_{ij}, corresponding to nine independent $J^{ij}_{\alpha\beta}$.

Without loss of generality, let us choose the following nine independent $S^q_{\alpha\beta}$:

$$
\begin{aligned}
S^a_{se} &= \frac{1}{2}\left(J^{02}_{se} - J^{13}_{se} - 2J^{23}_{se}\right) \; , \\
S^a_{s\tau} &= \frac{1}{2}\left(J^{02}_{s\tau} + J^{03}_{s\tau} - J^{23}_{s\tau}\right) \; , \\
S^a_{e\mu} &= \frac{1}{2}\left(-J^{12}_{e\mu} - J^{13}_{e\mu} - J^{23}_{e\mu}\right) \; , \\
S^b_{se} &= \frac{1}{2}\left(-J^{02}_{se} - J^{13}_{se}\right) \; , \\
S^b_{s\tau} &= \frac{1}{2}\left(-J^{02}_{s\tau} + J^{03}_{s\tau} + J^{23}_{s\tau}\right) \; , \\
S^b_{e\mu} &= \frac{1}{2}\left(J^{12}_{e\mu} - J^{13}_{e\mu} - J^{23}_{e\mu}\right) \; , \\
S^c_{se} &= \frac{1}{2}\left(J^{02}_{se} - J^{13}_{se}\right) \; , \\
S^c_{s\tau} &= \frac{1}{2}\left(J^{02}_{s\tau} + J^{03}_{s\tau} + J^{23}_{s\tau}\right) \; , \\
S^c_{e\mu} &= \frac{1}{2}\left(-J^{12}_{e\mu} - J^{13}_{e\mu} + J^{23}_{e\mu}\right) \; .
\end{aligned}
\tag{16}
$$

In terms of six flavor mixing angles and three independent phase combinations of V, we have expressed the nine independent $J^{ij}_{\alpha\beta}$ appearing on the right-hand side of Eq. (16) in Ref. [9]. Then one may directly obtain the explicit expressions of the nine-independent $S^q_{\alpha\beta}$ in terms of the same mixing angles and CP-violating phases. For illustration, we instructively take $s_{02}, s_{03}, s_{12}, s_{13} \sim \epsilon \ll 1$ [16]. In this approximate but simpler case, we arrive at

$$
\begin{aligned}
S^a_{se} &\approx \frac{1}{2}\left(-c_{01}s_{01}s_{02}s_{12}\sin\phi_z - c_{01}s_{01}s_{03}s_{13}\sin\phi_y\right) \ , \\
S^a_{s\tau} &\approx \frac{1}{2}\left[c_{23}s_{02}s_{03}s_{23}\sin\phi_x + c_{01}c^2_{23}s_{01}s_{03}s_{13}\sin\phi_y + c_{01}s_{01}s_{02}s_{12}s^2_{23}\sin\phi_z\right. \\
&\quad \left. + c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) - c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\right] \ , \\
S^a_{e\mu} &\approx -\frac{1}{2}\left[c_{01}c^2_{23}s_{01}s_{02}s_{12}\sin\phi_z + c_{01}s_{01}s_{03}s_{13}s^2_{23}\sin\phi_y + c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\right. \\
&\quad \left. - c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) + c_{23}s_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+\phi_z)\right] \ , \\
S^b_{se} &\approx \frac{1}{2}\left(c_{01}s_{01}s_{02}s_{12}\sin\phi_z - c_{01}s_{01}s_{03}s_{13}\sin\phi_y\right) \ , \\
S^b_{s\tau} &\approx \frac{1}{2}\left[(c^2_{01}-s^2_{01})c_{23}s_{02}s_{03}s_{23}\sin\phi_x + c_{01}c^2_{23}s_{01}s_{03}s_{13}\sin\phi_y - c_{01}s_{01}s_{02}s_{12}s^2_{23}\sin\phi_z\right. \\
&\quad \left. - c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) - c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\right] \ , \\
S^b_{e\mu} &\approx -\frac{1}{2}\left[-c_{01}c^2_{23}s_{01}s_{02}s_{12}\sin\phi_z + c_{01}s_{01}s_{03}s_{13}s^2_{23}\sin\phi_y - c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\right. \\
&\quad \left. - c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) + (s^2_{01}-c^2_{01})c_{23}s_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+\phi_z)\right] \ , \\
S^c_{se} &\approx \frac{1}{2}\left(-c_{01}s_{01}s_{02}s_{12}\sin\phi_z - c_{01}s_{01}s_{03}s_{13}\sin\phi_y\right) \ , \\
S^c_{s\tau} &\approx \frac{1}{2}\left[-c_{23}s_{02}s_{03}s_{23}\sin\phi_x + c_{01}c^2_{23}s_{01}s_{03}s_{13}\sin\phi_y + c_{01}s_{01}s_{02}s_{12}s^2_{23}\sin\phi_z\right. \\
&\quad \left. + c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) - c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\right] \ , \\
S^c_{e\mu} &\approx -\frac{1}{2}\left[c_{01}c^2_{23}s_{01}s_{02}s_{12}\sin\phi_z + c_{01}s_{01}s_{03}s_{13}s^2_{23}\sin\phi_y + c_{01}c_{23}s_{01}s_{03}s_{12}s_{23}\sin(\phi_x+\phi_z)\right. \\
&\quad \left. - c_{01}c_{23}s_{01}s_{02}s_{13}s_{23}\sin(\phi_x-\phi_y) - c_{23}s_{12}s_{13}s_{23}\sin(\phi_x-\phi_y+\phi_z)\right] \ ,
\end{aligned} \tag{17}
$$

where $\phi_x \equiv \delta_{03} - \delta_{02} - \delta_{23}$, $\phi_y \equiv \delta_{03} - \delta_{01} - \delta_{13}$ and $\phi_z \equiv \delta_{02} - \delta_{01} - \delta_{12}$. In obtaining these results, the corrections of $\mathcal{O}(\epsilon^3)$ or smaller have been neglected. Note that all $S^q_{\alpha\beta}$ given in Eq. (17) are suppressed by the factors of $\mathcal{O}(\epsilon^2)$. Note also that $S^a_{se} \approx S^c_{se}$ holds, as a consequence of $J^{23}_{se} \approx 0$ in the approximation made above.

As some direct relations between the rephasing invariants of CP violation $J^{ij}_{\alpha\beta}$ and the probability asymmetries of neutrino oscillations $\Delta P_{\alpha\beta}$ have been established in Ref. [9], one can straightforwardly obtain the relations between $\Delta_{\alpha\beta}$ and $S^q_{\alpha\beta}$ or S^q_{ij} with the help of Eqs. (9) and (10). For simplicity, we do not go into detail at this point. We shall not discuss possible terrestrial matter effects on the unitarity quadrangles in realistic long-baseline neutrino oscillation experiments either, because the relevant discussions given in Refs. [9,17] are completely applicable here. Furthermore, we point out that the present experimental constraints on the 4×4 neutrino mixing matrix V remain too poor to reveal its structural features (such as possible symmetries or asymmetries of its off-diagonal matrix elements [18]).

In summary, we have presented a concise classification for unitarity quadrangles of the 4×4 neutrino mixing matrix. It is found that there are totally thirty-six distinct topologies among twelve different unitarity quadrangles. Useful relations between the areas of those unitarity quadrangles and the rephasing invariants of CP violation have been derived. For illustration, we have also expressed nine independent areas of the unitarity quadrangles approximately in terms of six flavor mixing angles and three CP-violating phases in the standard parametrization.

Finally, we remark that our analytical results are model-independent and would be very useful for a systematic study of CP and T violation in a variety of long-baseline neutrino oscillation experiments, if the forthcoming MiniBooNE experiment could confirm the LSND anomaly and support the scheme of four neutrino mixing.

This work was supported in part by the National Natural Science Foundation of China.

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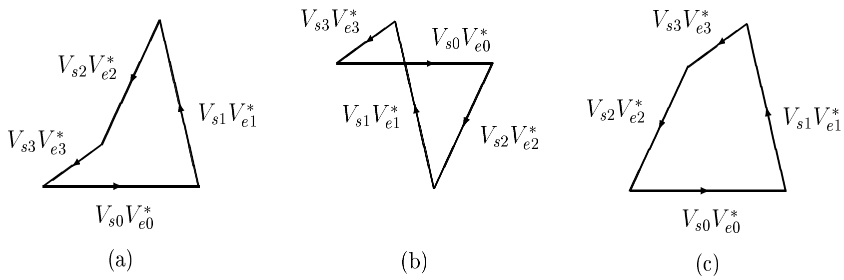
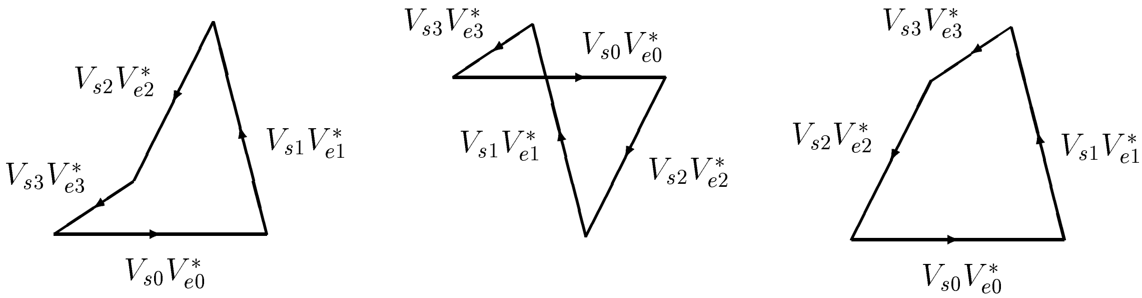


FIG. 1. Three distinct topologies of unitarity quadrangle Q_{se} .


$V_{s2}V_{e2}^*$
$V_{s1}V_{e1}^*$
$V_{s3}V_{e3}^*$
$V_{s0}V_{e0}^*$
$V_{s3}V_{e3}^*$
$V_{s0}V_{e0}^*$
$V_{s1}V_{e1}^*$
$V_{s2}V_{e2}^*$
$V_{s3}V_{e3}^*$
$V_{s2}V_{e2}^*$
$V_{s1}V_{e1}^*$
$V_{s0}V_{e0}^*$

(a) (b) (c)